

A Well-Founded Semantics for FOL-Programs

Yi Bi¹, Jia-Huai You², Zhiyong Feng¹

¹Tianjin University, Tianjin China

²University of Alberta, Edmonton T6G 2E8, Canada

submitted 1 January 2003; revised 1 January 2003; accepted 1 January 2003

Abstract

An FOL-program consists of a background theory in a decidable fragment of first-order logic and a collection of rules possibly containing first-order formulas. The formalism stems from recent approaches to tight integrations of ASP with description logics. In this paper, we define a well-founded semantics for FOL-programs based on a new notion of unfounded sets on consistent as well as inconsistent sets of literals, and study some of its properties. The semantics is defined for all FOL-programs, including those where it is necessary to represent inconsistencies explicitly. The semantics supports a form of combined reasoning by rules under closed world as well as open world assumptions, and it is a generalization of the standard well-founded semantics for normal logic programs. We also show that the well-founded semantics defined here approximates the well-supported answer set semantics for normal DL programs.

To appear in *Theory and Practice of Logic Programming (TPLP)*.

KEYWORDS: Logic Programs, Well-Founded Semantics, First-Order Logic.

1 Introduction

Recent literature has shown extensive interests in combining ASP with fragments of classical logic, such as description logics (DLs) (see, e.g., (de Bruijn et al. 2008; de Bruijn et al. 2007; Eiter et al. 2008; Lukasiewicz 2010; Motik and Rosati 2010; Rosati 2005; Rosati 2006; Shen and Wang 2011; Yang et al. 2011)). A program in this context is a combined knowledge base $KB = (L, \Pi)$, where L is a knowledge base of a decidable fragment of first-order logic and Π a set of rules possibly containing first-order formulas or interface facilities. In this paper, we use *FOL-program* as an umbrella term for approaches that allow first-order formulas to appear in rules (the so-called *tight integration*), for generality. The goal of this paper is to formulate a well-founded semantics for these programs with the following features.

- The class of all FOL-programs are supported.
- Combined reasoning with closed world as well as open world assumptions is supported.

Under the first feature, we shall allow an atom with its predicate shared in the first-order theory L to appear in a rule head. This can result in two-way flow of information and enable inferences within each component automatically. For example, assume L contains a formula that says students are entitled to educational discount, $\forall x St(x) \supset EdDiscount(x)$. Using the notation of DL, we would write $St \sqsubseteq EdDiscount$. Suppose in an application anyone who is not employed full time but registered for an evening class is given the benefit of a student. We can write a rule

$$St(X) \leftarrow EveningClass(X), not HasJob(X).$$

Thus, that such a person enjoys educational discount can be inferred directly from the underlying knowledge base L .

To support all FOL-programs, we need to consider the possibility of inconsistencies arising in the construction of the intended well-founded semantics. For example, consider an FOL-program, $KB = (L, \Pi)$, where $L = \{\forall x A(x) \supset C(x), \neg C(a)\}$ and $\Pi = \{A(a) \leftarrow \text{not } B(a); B(a) \leftarrow B(a)\}$. Suppose the Herbrand base is $\{A(a), B(a)\}$. In an attempt to compute the well-founded semantics of KB by an iterative process, we begin with the empty set; while L entails $\neg A(a)$, since $B(a)$ is false by closed world reasoning, we derive $A(a)$ resulting in an inconsistency. This reasoning process suggests that during an iterative process a consistent set of literals may be mapped to an inconsistent one and, in general, whether inconsistencies arise or not is not known *a priori* without actually performing the computation.

That the well-founded semantics of an FOL-program is defined by an inconsistent set can be useful on its own, or in the computation of (suitably defined) answer sets of the program. If we have computed the well-founded semantics which is inconsistent, we need not pursue the task of computing answer sets of the same program, because they do not exist.

In complex real world reasoning by rules, it is sometimes desirable that not all predicates are reasoned with under the closed world assumption. Some conditions may need to be established under the open world assumption. We call this *combined reasoning*. For example, we may write a rule

$$PrescribeTo(X, Q) \leftarrow Effective(X, Z), Contract(Q, Z), \neg AllergicTo(Q, X)$$

to describe that an antibiotic is prescribed to a patient who contracted a bacterium, if the antibiotic against that bacterium is effective and patient is not allergic to it. Though *Effective* can be reasoned with under the closed world assumption, it may be preferred to judge whether a patient is not allergic to an antibiotic under the open world assumption, e.g., it holds if it can be proved classically. This is in contrast with closed world reasoning whereas one may infer nonallergic due to lack of evidence for allergy.

To our knowledge, there has been no well-founded semantics defined for all FOL-programs. The closest that one can find is the definition for a subset of FOL-programs (Lukasiewicz 2010), which relies on syntactic restrictions so that the least fixpoint is computed over consistent sets of literals. To ensure that the construction is well-defined, it is assumed that DL axioms must be, or can be converted to, *tuple generating dependencies* (which are essentially Horn rules) plus constraints. Thus, the approach cannot be lifted to handle first-order formulas in general. In addition, to the best of our knowledge, no combined reasoning is ever supported under any well-founded semantics.

As motivated above, in this paper we first define a well-founded semantics for FOL-programs based on a new notion of unfounded sets. We show that the semantics generalizes the well-founded semantics for normal logic programs. Also, we prove that the well-founded semantics defined here approximates the well-supported answer set semantics for the language of (Shen and Wang 2011); namely, all well-founded atoms (resp. unfounded atoms) of a program remain to be true (resp. false) in any well-supported answer set. This makes it possible to use the mechanism of constructing the well-founded semantics as constraint propagation in an implementation of computing well-supported answer sets.

The paper is organized as follows. The next section introduces the language and notations. In Section 3 we define the well-founded semantics. Section 4 studies some properties and relates

to the well-supported answer set semantics, followed by related work in Section 5. Section 6 concludes the paper and points to future directions.

2 Language and Notation

We assume a language of a decidable fragment of first-order logic, denoted \mathcal{L}_Σ , where $\Sigma = \langle F^n; P^n \rangle$, called a signature, and F^n and P^n are disjoint countable sets of n -ary function and n -ary predicate symbols, respectively. Constants are 0-ary functions. *Terms* are variables, constants, or functions in the form $f(t_1, \dots, t_n)$, where each t_i is a term and $f \in F^n$. *First-order formulas*, or just *formulas*, are defined as usual, so are the notions of *satisfaction*, *model*, and *entailment*.

Let Φ_P be a finite set of predicate symbols and Φ_C a nonempty finite set of constants such that $\Phi_C \subseteq F^n$. An *atom* is of the form $P(t_1, \dots, t_n)$ where $P \in \Phi_P$ and each t_i is either a constant from Φ_C or a variable. A *negated atom* is of the form $\neg A$ where A is an atom. We do not assume any other restriction on the vocabularies, that is, Φ_P and P^n may have predicate symbols in common.

An *FOL-program* is a combined knowledge base $KB = (L, \Pi)$, where L is a first-order theory of \mathcal{L}_Σ and Π a *rule base*, which is a finite collection of rules of the form

$$H \leftarrow A_1, \dots, A_m, \text{not } B_1, \dots, \text{not } B_n \quad (1)$$

where H is an atom, and A_i and B_i are atoms or formulas. By abuse of terminology, each A_i is called a *positive literal* and each $\text{not } B_i$ is called a *negative literal*.

For any rule r , we denote by $\text{head}(r)$ the head of the rule and $\text{body}(r)$ its body, and we define $\text{pos}(r) = \{A_1, \dots, A_m\}$ and $\text{neg}(r) = \{B_1, \dots, B_n\}$.

A *ground instance* of a rule r in Π is obtained by replacing every free variable with a constant from Φ_C . In this paper, we assume that a rule base Π is already grounded if not said otherwise. When we refer to an atom/literal/formula, by default we mean it is a ground one.

Given an FOL-program $KB = (L, \Pi)$, the *Herbrand base* of Π , denoted HB_Π , is the set of all ground atoms $P(t_1, \dots, t_n)$, where $P \in \Phi_P$ occurs in KB and $t_i \in \Phi_C$.

We denote by Ω the set of all predicate symbols appearing in HB_Π such that $\Omega \subseteq P^n$. For distinction, we call atoms whose predicate symbols are not in Ω *ordinary*, and all the other formulas *FOL-formulas*. If $L = \emptyset$ and Π only contains rules of the form (1) where all H , A_i and B_j are ordinary atoms, then KB is called a *normal logic program*.

Any subset $I \subseteq HB_\Pi$ is called an *interpretation* of Π . It is also called a *total* interpretation or a *2-valued* interpretation. If I is an interpretation, we define $\bar{I} = HB_\Pi \setminus I$.

Let Q be a set of atoms. We define $\neg.Q = \{\neg A \mid A \in Q\}$. For a set of atoms and negated atoms S , we define $S^+ = \{A \mid A \in S\}$, $S^- = \{A \mid \neg A \in S\}$, and $S|_\Omega = \{A \in S \mid \text{pred}(A) \in \Omega\}$, where $\text{pred}(A)$ is the predicate symbol of A . Let $\text{Lit}_\Pi = HB_\Pi \cup \neg.HB_\Pi$. A subset $S \subseteq \text{Lit}_\Pi$ is consistent if $S^+ \cap S^- = \emptyset$. For a first-order theory L , we say that $S \subseteq \text{Lit}_\Pi$ is *consistent with L* if the first-order theory $L \cup S|_\Omega$ is consistent (i.e., the theory is *satisfiable*). Note that when we say S is consistent with L , both S and L must be consistent. Similarly, a (2-valued) interpretation I is consistent with L if $L \cup I|_\Omega \cup \neg.\bar{I}|_\Omega$ is consistent. We denote by Lit_Π^c the set of all consistent subsets of Lit_Π . For any $S \in \text{Lit}_\Pi^c$, S' is called a *consistent extension* of S if $S \subseteq S' \in \text{Lit}_\Pi^c$.

Definition 1

Let $KB = (L, \Pi)$ be an FOL-program and $I \subseteq HB_\Pi$ an interpretation. Define the satisfaction relation under L , denoted \models_L , as follows (the definition extends to conjunctions of literals):

1. For any ordinary atom $A \in HB_{\Pi}$, $I \models_L A$ if $A \in I$ and $I \models \text{not } A$ if $A \notin I$.
2. For any FOL-formula A , $I \models_L A$ if $L \cup I|_{\Omega} \cup \neg \bar{I}|_{\Omega} \models A$, and $I \models_L \text{not } A$ if $I \not\models_L A$.

Let $KB = (L, \Pi)$ be an FOL-program. For any $r \in \Pi$ and $I \subseteq HB_{\Pi}$, $I \models_L r$ if $I \not\models_L \text{body}(r)$ or $I \models_L \text{head}(r)$. I is a *model* of KB if I is consistent with L and I satisfies all rules in Π .

Example 1

To illustrate the flexibility provided by the parameter Ω , suppose we have a program $KB = (L, \Pi)$ where Π contains a rule that says any unemployed with disability receives financial assistance, with an FOL-formula in the body

$$\text{Assist}(X) \leftarrow \text{Disabled}(X), \text{not } \text{Employed}(X)$$

Assume $\Omega = \Phi_P = \{\text{Assist}, \text{Employed}\}$. Then, *Employed* is interpreted under the closed world assumption and *Disabled* under the open world assumption. Indeed, unemployment can be established by closed world reasoning for lack of evidence of employment, but disability requires a direct proof.

3 Well-Founded Semantics

We first define the notion of unfounded set. Intuitively, the atoms in an unfounded set can be safely assigned to false, due to persistent inability to derive their positive counterparts.

Definition 2

(Unfounded set) Let $KB = (L, \Pi)$ be an FOL-program and $I \subseteq Lit_{\Pi}$. If $I \cup L$ is consistent, then a set $U \subseteq HB_{\Pi}$ is an *unfounded set* of KB relative to I iff for every $H \in U$ and $r \in \Pi$, both of the following conditions are satisfied

- (a) If $\text{head}(r) = H$, then
 - (i) $\neg A \in I \cup \neg.U$ for some ordinary atom $A \in \text{pos}(r)$, or
 - (ii) $B \in I$ for some ordinary atom $B \in \text{neg}(r)$, or
 - (iii) for some FOL-formula $A \in \text{pos}(r)$, it holds that $L \cup S|_{\Omega} \not\models A$, for all $S \in Lit_{\Pi}^c$ with $I \cup \neg.U \subseteq S$, or
 - (iv) for some FOL-formula $A \in \text{neg}(r)$, $L \cup I|_{\Omega} \models A$.
- (b) $L \cup S|_{\Omega} \not\models H$ for all $S \in Lit_{\Pi}^c$ with $I \cup \neg.U \subseteq S$.

If $I \cup L$ is inconsistent, the unfounded set of KB relative to I is HB_{Π} .

That H is unfounded relative to I if both conditions (a) and (b) are satisfied when $I \cup L$ is consistent; in particular, condition (a.iii) ensures that a positive occurrence of an FOL-formula in the rule body is not entailed, for all consistent extensions of $I \cup \neg.U$; and condition (b) ensures the inability to infer its positive counterpart, independent of any rules.

An FOL-formula may contain shared predicates in Ω , and those not in Ω hence not shared. The latter are supposed to be interpreted under open world assumption. Continuing with Example 1 above, let $KB = (L, \Pi)$, where

$$\begin{aligned} L &= \{\forall x \text{Certified}(x) \supset \text{Disabled}(x)\} \\ \Pi &= \{\text{Assist}(a) \leftarrow \text{Disabled}(a), \text{not } \text{Employed}(a)\} \end{aligned}$$

Assume $\text{Assist}, \text{Employed} \in \Omega$ while *Certified* and *Disabled* are not. Let $\Phi_C = \{a\}$, and thus $HB_{\Pi} = \{\text{Assist}(a), \text{Employed}(a)\}$. Clearly, $\{\text{Assist}(a), \text{Employed}(a)\}$ is an unfounded

set relative to $I = \emptyset$, in particular because $Disabled(a)$ is not derivable under all consistent extensions of I . Note that, since $Disabled(a)$ is not in HB_{Π} , it is not part of an unfounded set.¹

Lemma 1

Let $KB = (L, \Pi)$ be an FOL-program and $I \subseteq Lit_{\Pi}$. A set of unfounded sets of KB relative to I is closed under union, and the greatest unfounded set of KB relative to I exists, which is the union of all unfounded sets of KB relative to I .

Proof

If I is inconsistent, the claims hold trivially. For a consistent I , suppose $U_1, U_2 \subseteq HB_{\Pi}$ are unfounded sets (of KB relative to I), we show that $U_1 \cup U_2$ is also an unfounded set. Let $A \in U_1$. Since both (a) and (b) in Definition 2 hold for U_1 and U_2 separately, in particular, each consistent extension of $I \cup \neg.(U_1 \cup U_2)$ is a consistent extension of $I \cup \neg.U_1$, (a) and (b) also hold for $U_1 \cup U_2$. Thus $A \in U_1 \cup U_2$. By symmetry, the same argument applies to U_2 . Therefore, the union of all unfounded sets is an unfounded set, which is the greatest among all unfounded sets. \square

We define the operators which will be used to define the well-founded semantics.

Definition 3

Let $KB = (L, \Pi)$ be an FOL-program. Define T_{KB}, U_{KB}, Z_{KB} as mappings of $2^{Lit_{\Pi}} \rightarrow HB_{\Pi}$, and W_{KB} as a mapping of $2^{Lit_{\Pi}} \rightarrow 2^{Lit_{\Pi}}$, as follows:

- (i) If $I \cup L$ is inconsistent, then $T_{KB}(I) = HB_{\Pi}$; otherwise, $H \in T_{KB}(I)$ iff $H \in HB_{\Pi}$ and either (a) or (b) below holds
 - (a) some $r \in \Pi$ with $head(r) = H$ exists such that
 - (1) for any ordinary atom A , $A \in I$ if $A \in pos(r)$ and $\neg A \in I$ if $A \in neg(r)$,
 - (2) for any FOL-formula $A \in pos(r)$, $L \cup I|_{\Omega} \models A$, and
 - (3) for any FOL-formula $B \in neg(r)$, $L \cup S|_{\Omega} \not\models B$, for all $S \in Lit_{\Pi}^c$ with $I \subseteq S$.
 - (b) $L \cup I|_{\Omega} \models H$.
- (ii) $U_{KB}(I)$ is the greatest unfounded set of KB relative to I .
- (iii) $Z_{KB}(I) = \{A \in HB_{\Pi} \mid L \cup I|_{\Omega} \models \neg A\}$.
- (iv) $W_{KB}(I) = T_{KB}(I) \cup \neg.U_{KB}(I) \cup \neg.Z_{KB}(I)$.

The operator T_{KB} is a consequence operator. An atom is a consequence, either due to a derivation via a rule (case (i.a)), or because it is entailed by L , given I (case (i.b)). In the first case, the body of such a rule should be satisfied not only by I , but by all consistent extensions of I . In the case (i.a.1) or (i.a.2), it is sufficient that the body is satisfied by I only because the classical entailment relation is monotonic. For the case (i.a.3) the condition needs to be stated explicitly.

There are two features in this definition that are non-conventional. The first is the operator Z_{KB} - interacting an FOL knowledge base with rules may result in *direct negative consequences*. In the second, all operators here are defined on all subsets of Lit_{Π} , including inconsistent ones.²

Lemma 2

The operators T_{KB}, U_{KB}, Z_{KB} , and W_{KB} are all monotonic.

¹ Placed under the context of 2-valued logic, the reasoning here is analogue to parallel circumscription (McCarthy 1980), where the predicates *Employed* and *Assist* are minimized with predicates *Certified* and *Disabled* varying.

² When inconsistency arises, the fixpoint operator here leads to triviality. This is the most common treatment when the underlying entailment relation is the classical one. However, we remark that this is only one possible choice.

Proof

Let $I_1 \subseteq I_2 \subseteq Lit_\Pi$ and $H \in T_{KB}(I_1)$. Since any condition in part (i) of Definition 3 that applies under I_1 applies under I_2 (including the case where one of them, or both, are inconsistent with L), thus the set of all consistent extensions of I_2 is a subset of all consistent extensions of I_1 , and therefore we have $T_{KB}(I_1) \subseteq T_{KB}(I_2)$. The same argument applies to U_{KB} and Z_{KB} . Since T_{KB} , U_{KB} , and Z_{KB} are monotonic, it follows from the definition that the operator W_{KB} is also monotonic. \square

As W_{KB} is monotone on the complete lattice $\langle 2^{Lit_\Pi}, \subseteq \rangle$, according to the Knaster-Tarski fix-point theorem (Tarski 1955), its least fixpoint, $lfp(W_{KB})$, exists.

Definition 4

Let $KB = (L, \Pi)$ be an FOL-program. The *well-founded semantics* of KB (relative to Ω) is defined by $lfp(W_{KB})$.

We allow the well-founded semantics of an FOL-program to be defined by an inconsistent set, independent of how the semantics may be used. This may be utilized in the computation of answer sets. Suppose under a suitable definition of answer sets for an FOL-program KB , $lfp(W_{KB})$ approximates all answer sets of KB .³ If the computed $lfp(W_{KB})$ is inconsistent then we know KB has no answer sets.

Example 2

Let $KB = (\{\neg A(a)\}, \Pi)$ where $\Pi = \{A(a) \leftarrow not\ B(a), B(a) \leftarrow B(a)\}$. Let $\Omega = \Phi_P = \{A, B\}$ and $\Phi_C = \{a\}$. $lfp(W_{KB})$ is constructed as follows (where $W_{KB}^0 = \emptyset$ and $W_{KB}^{i+1} = W_{KB}(W_{KB}^i)$, for all $i \geq 0$):

$$\begin{aligned} W_{KB}^0 &= \emptyset, \\ W_{KB}^1 &= \{\neg A(a), \neg B(a)\}, \\ W_{KB}^2 &= \{\neg A(a), \neg B(a), A(a)\}, \\ W_{KB}^3 &= Lit_\Pi, \\ W_{KB}^4 &= W_{KB}^3. \end{aligned}$$

As a result, the well-founded semantics of KB is inconsistent. It is interesting to note that KB has a model, $\{B(a)\}$. This means that we cannot determine whether the well-founded semantics for an FOL-program is consistent or not, based on the existence of a model; when an iterative process is carried out, we have to deal with the possibility that inconsistencies may arise.

Example 3

Consider $KB = (L, \Pi)$ where $L = \{\forall x B(x) \supset A(x), \neg A(a) \vee C(a)\}$ and Π consists of

$$\begin{aligned} B(a) &\leftarrow B(a). \\ A(a) &\leftarrow (\neg C(a) \wedge B(a)). \\ R(a) &\leftarrow not\ C(a), not\ A(a). \end{aligned}$$

Let $\Phi_P = \{A, B, R\}$, $\Omega = \{A, B\}$, and $\Phi_C = \{a\}$. Hence $HB_\Pi = \{A(a), B(a), R(a)\}$. For $I_0 = \emptyset$, we have $T_{KB}(I_0) = \emptyset$, $U_{KB}(I_0) = \{B(a), A(a)\}$, and $Z_{KB} = \emptyset$. $B(a)$ is in $U_{KB}(I_0)$ because $B(a)$ is not derivable by any rule based on I_0 , and $L \cup S|_\Omega \not\models B(a)$ for all $S \in Lit_\Pi^c$ with $I_0 \cup \neg U_{KB}(I_0) \subseteq S$ (condition (b) in Definition 2). Similarly, $A(a)$ is in $U_{KB}(I_0)$. Since

³ We will show later in this paper that the well-supported answer sets of (Shen and Wang 2011) fall into this category.

$C(a)$ is not derivable under all consistent extensions, we derive $R(a)$. Therefore, $\text{lfp}(W_{KB}) = \{\neg B(a), \neg A(a), R(a)\}$. Note that since $C(a) \notin HB_{\Pi}$ its truth value is not part of the well-founded semantics.

4 Properties and Relations

We first show that the well-founded semantics is a generalization of the well-founded semantics for normal logic programs.

Theorem 1

Let $KB = (\emptyset, \Pi)$ be a normal logic program. Then, the WFS of KB coincides with the WFS of Π .

Proof

The WFS of a normal program Π is defined by the least fixpoint of a monotone operator W_{Π} on the set of consistent subsets of $HB_{\Pi} \cup \neg HB_{\Pi}$:

- $T_{\Pi}(S) = \{\text{head}(r) \mid r \in \Pi, \text{pos}(r) \cup \neg \text{neg}(r) \subseteq S\}$
- $W_{\Pi}(S) = T_{\Pi}(S) \cup \neg U_{\Pi}(S)$

where $U_{\Pi}(S)$ is the greatest unfounded set of Π w.r.t. S . A set $U \subseteq HB_{\Pi}$ is an unfounded set of Π w.r.t. S , if for every $a \in U$ and every rule $r \in \Pi$ with $\text{head}(r) = a$, either (i) $\neg b \in S \cup \neg U$ for some $b \in \text{pos}(r)$, or (ii) $b \in S$ for some atom $b \in \text{neg}(r)$.

Then, it is immediate that the notion of unfounded set and the greatest unfounded set for normal logic program $KB = (\emptyset, \Pi)$ coincide with those for Π , respectively. Note that $Z_{KB}(I) \subseteq U_{KB}(I)$ when $L = \emptyset$. It is easy to see that the operator T_{KB} for normal program $KB = (\emptyset, \Pi)$ reduces to T_{Π} for normal program Π . \square

The well-supported answer set semantics is defined for what are called *normal DL logic programs* (Shen and Wang 2011), which applies to FOL-programs. There is however a subtle difference: in the definition of the entailment relation, the W_{KB} operator uses 3-valued evaluation while the well-supported semantics is based on the notion of 2-valued *up to satisfaction*.

Definition 5

(Up to satisfaction) Let $KB = (L, \Pi)$, l a literal, and E and I two interpretations with $E \subseteq I \subseteq HB_{\Pi}$. The relation E up to I satisfies l under L , denoted $(E, I) \models_L l$, is defined as: $(E, I) \models_L l$ if $\forall F, E \subseteq F \subseteq I, F \models_L l$. The definition extends to conjunctions of literals.

The entailment relation, $F \models_L l$, is based 2-valued satisfiability (cf. Def. 1), i.e., $F \models_L l$ is $L \cup F|_{\Omega} \cup \neg \bar{F}|_{\Omega} \models l$, while in 3-valued satisfiability, $S \models_L l$ is $L \cup S|_{\Omega} \models l$.

Given an FOL-program $KB = (L, \Pi)$, an immediate consequence operator is defined as:

$$\mathcal{T}_{KB}(E, I) = \{\text{head}(r) \mid r \in \Pi, (E, I) \models_L \text{body}(r)\}. \quad (2)$$

The operator \mathcal{T}_{KB} is monotonic on its first argument E with I fixed (Shen and Wang 2011). Thus, for any model I of KB , we can compute a fixpoint, denoted $\mathcal{T}_{KB}^{\alpha}(\emptyset, I)$.

Definition 6

Let $KB = (L, \Pi)$ be an FOL-program and I a model of KB . I is an *answer set of KB* if for every $A \in I$, either $A \in \mathcal{T}_{KB}^{\alpha}(\emptyset, I)$ or $L \cup \mathcal{T}_{KB}^{\alpha}(\emptyset, I)|_{\Omega} \cup \neg \bar{I}|_{\Omega} \models A$.

The next theorem shows that the well-founded semantics of an FOL-program approximates its well-supported answer set semantics.

Theorem 2

Let $KB = (L, \Pi)$ be an FOL-program. Then every well-supported answer set of KB includes all atoms $H \in HB_\Pi$ that are well-founded and no atoms $H \in HB_\Pi$ that are unfounded or are in $Z_{KB}(lfp(W_{KB}))$.

Proof

To prove the assertion, it is sufficient to show that if $lfp(W_{KB})$ is consistent, then for every well-supported answer set I , all atoms in $lfp(W_{KB})$ are in I and all negated atoms in $lfp(W_{KB})$ are in $\neg \bar{I}$.

We consider the fixpoint construction by the operators $\mathcal{T}_{KB}(\cdot, I)$ and W_{KB} . Let us use a short notation for the respective sequences by

$$\mathcal{T}_{KB}^0 = \emptyset, \dots, \mathcal{T}_{KB}^{k+1} = \mathcal{T}_{KB}(\mathcal{T}_{KB}^k, I), \dots \quad (3)$$

$$W_{KB}^0 = \emptyset, \dots, W_{KB}^{k+1} = W(W_{KB}^k), \dots \quad (4)$$

for all $k \geq 0$. Define $E_0 = \emptyset$ and $E_i = \{H \mid (\mathcal{T}_{KB}^{i-1}, I) \models_L H\}$ for all $i \geq 1$. We show that $W_{KB}^i \subseteq E_i \cup \mathcal{T}_{KB}^i \cup \neg \bar{I}$, for all $i \geq 0$. The base case is trivial. For the inductive step, assume for any $k \geq 0$ the subset relation holds and we show it holds for $k+1$. The proof is conducted on two cases: **(I)** Assume an atom $H \in W_{KB}^{k+1}$ and show $H \in E_{k+1} \cup \mathcal{T}_{KB}^{k+1}$, and **(II)** Assume a negated atom $\neg H \in W_{KB}^{k+1}$ and show $\neg H \in \neg \bar{I}$.

By definition and monotonicity of the operator \mathcal{T}_{KB} , $(\mathcal{T}_{KB}^i, I) \models_L E_i$, and it follows from the first-order entailment that for any atom $H \in HB_\Pi$,

$$(\mathcal{T}_{KB}^i, I) \models_L H \text{ iff } (E_i \cup \mathcal{T}_{KB}^i, I) \models_L H. \quad (5)$$

By definition,

$$W_{KB}^{k+1} = T_{KB}(W_{KB}^k) \cup \neg \cdot U_{KB}(W_{KB}^k) \cup \neg \cdot Z_{KB}(W_{KB}^k)$$

By Proposition 1 of (Shen 2011), for any FOL-formula H ,

$$(E, I) \models_L H \text{ iff } L \cup E|_\Omega \cup \neg \bar{I}|_\Omega \models H. \quad (6)$$

By Proposition 2 (Shen 2011), for any ordinary atom H ,

$$(E, I) \models_L H \text{ iff } H \in E; (E, I) \models_L \text{not } H \text{ iff } H \notin I. \quad (7)$$

(I) For any atom $H \in W_{KB}^{k+1}$, we have $H \in T_{KB}(W_{KB}^k)$. If condition (i.b) in Definition 3 holds, we have $L \cup W_{KB}^k|_\Omega \models H$. By induction hypothesis, $L \cup (E_k \cup \mathcal{T}_{KB}^k)|_\Omega \cup \neg \bar{I}|_\Omega \models H$. Then by (6) and (5), $(\mathcal{T}_{KB}^k, I) \models_L H$. Thus $H \in E_{k+1}$. If condition (i.a) in Definition 3 holds, we consider the following four cases:

1. For any ordinary atom $A \in pos(r)$, $A \in W_{KB}^k$, thus $(\mathcal{T}_{KB}^k, I) \models_L A$ by (7) and (5).
2. For any ordinary atom $A \in neg(r)$, $\neg A \in W_{KB}^k$, thus $(\mathcal{T}_{KB}^k, I) \models_L \text{not } A$.
3. For any FOL-formula $A \in pos(r)$, $L \cup W_{KB}^k|_\Omega \models A$, then $(\mathcal{T}_{KB}^k, I) \models_L A$.
4. For any FOL-formula $A \in neg(r)$, $L \cup (W_{KB}^k)'|_\Omega \not\models A$ for every $(W_{KB}^k)'$ such that $W_{KB}^k \subseteq (W_{KB}^k)' \in Lit_\Pi^c$, we have $(\mathcal{T}_{KB}^k, I) \not\models_L A$, since every total interpretation is a partial one.

Hence $H \in \mathcal{T}_{KB}^{k+1}$.

(II) For any negated atom $\neg H \in W_{KB}^{k+1}$, either $H \in U_{KB}(W_{KB}^k)$ or $H \in Z_{KB}(W_{KB}^k)$. For the case $H \in U_{KB}(W_{KB}^k)$, if condition (b) in Definition 2 holds, we have $(\mathcal{T}_{KB}^k, I) \not\models_L H$ by (5), (6) and induction hypothesis, in addition to the fact that every total interpretation is a partial one. Then $H \notin E_{k+1}$. For condition (a) in Definition 2, we also consider the following four situations:

1. For any ordinary atom $A \in pos(r)$, $\neg A \in W_{KB}^k \cup \neg U$, thus $(\mathcal{T}_{KB}^k, I) \not\models_L A$, by a similar argument above.
2. For any ordinary atom $A \in neg(r)$, $A \in W_{KB}^k$, thus $(\mathcal{T}_{KB}^k, I) \models_L A$.
3. For any FOL-formula $A \in pos(r)$, $L \cup (W_{KB}^k)'|_{\Omega} \not\models A$, for every $W_{KB}^k \subseteq (W_{KB}^k)' \in Lit_{II}^c$, then $(\mathcal{T}_{KB}^k, I) \not\models_L A$, since every total interpretation is a partial one.
4. For any FOL-formula $A \in neg(r)$, $L \cup W_{KB}^k|_{\Omega} \models A$, we have $(\mathcal{T}_{KB}^k, I) \models_L A$.

We have $H \notin \mathcal{T}_{KB}^{k+1}$. Hence $H \notin E_{k+1} \cup \mathcal{T}_{KB}^{k+1}$

For any $\neg H \in W_{KB}^k$, $\neg H \in \neg \bar{I}$, since the operator E and \mathcal{T}_{KB} only generate positive atoms. We then have $H \notin E_k \cup \mathcal{T}_{KB}^k$. As k is arbitrary, we have $H \in U_{KB}(lfp(W_{KB}))$ and $H \notin E_{\alpha} \cup \mathcal{T}_{KB}^{\alpha}$, where E_{α} and $\mathcal{T}_{KB}^{\alpha}$ are the respective fixpoints. Since $E_{\alpha} \cup \mathcal{T}_{KB}^{\alpha} = I$ (By definition 6), we get $H \in \bar{I}$. Similarly, if $H \in Z_{KB}(W_{KB}^k)$, then $H \in \bar{I}$. We thus have proved that $W_{KB}^{k+1} \subseteq E_{k+1} \cup \mathcal{T}_{KB}^{k+1} \cup \neg \bar{I}$. \square

5 Related Work

The most relevant work in defining well-founded semantics for combining rules with DLs are (Eiter et al. 2011; Lukasiewicz 2010). The former embeds *dl-atoms* in rule bodies to serve as queries to the underlying ontology, and it does not allow the predicate in a rule head to be shared in the ontology. In both approaches, syntactic restrictions are posted so that the least fixpoint is always constructed over sets of consistent literals. It is also a unique feature in our approach that combined reasoning with closed world and open world is supported.

A program in FO(ID) has a clear knowledge representation “task” - the rule component is used to define concepts, whereas the FO component may assert additional properties of the defined concepts. All formulas in FO(ID) are interpreted under closed world assumption. Thus, FOL-programs and FO(ID) have fundamental differences in basic ideas. On semantics, FOL-formulas can be interpreted under open world and closed world flexibly. On modeling, the rule set in FO(ID) is built on ontologies, thus information can only flow from a first order theory to rules. But in FOL-programs, the first order theory and rules are tightly integrated, and thus information can flow from each other bilaterally.

6 Conclusion and Future Directions

In this paper we have defined a new well-founded semantics for FOL-programs, where arbitrary FOL-formulas are allowed to appear in rule bodies and an atom with its predicate shared with first-order theory to appear in a rule head. Combined reasoning with closed world as well as open world is supported. Moreover, inconsistencies are dealt with explicitly, and thus the task of computing answer sets can be prejudged in case that the well-founded semantics is an inconsistent set. We have shown that the well-founded semantics is an appropriate approximation of the well-supported answer set semantics defined in (Shen and Wang 2011).

As future work, we will study the approximation fixpoint theory (AFT) (Denecker et al. 2000; Denecker et al. 2004), and investigate whether and how well-founded and stable semantics of FOL-programs can be defined uniformly under an extended approximation fixpoint theory. We are also interested in possible different approximating operators for alternative semantics of FOL-programs. In (Denecker et al. 2004) the authors show that the theory of consistent approximations can be applied to the entire bilattice \mathcal{L}^2 (including inconsistent elements), under the

assumption that an approximating operator \mathcal{A} is *symmetric*. This symmetry assumption guarantees that no transition from a consistent state to an inconsistent one may take place. As argued at the outset of this paper, this is precisely what we cannot assume for a definition of well-founded semantics for all FOL-programs.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (NSFC) grants 61373035 and 61373165, and by National High-tech R&D Program of China (863 Program) grant 2013AA013204.

References

- DE BRUIJN, J., EITER, T., AND TOMPITS, H. 2008. Embedding approaches to combining rules and ontologies into autoepistemic logic. In *Proc. KR 2008*. 485–495.
- DE BRUIJN, J., PEARCE, D., POLLERES, A., AND VALVERDE, A. 2007. Quantified equilibrium logic and hybrid rules. In *Proc. RR 2007*. 58–72.
- DENECKER, M., MAREK, V., AND TRUSZCZYŃSKI, M. 2000. Approximations, stable operators, well-founded fixpoints and applications in nonmonotonic reasoning. In *Logic-based Artificial Intelligence*. 127–144.
- DENECKER, M., MAREK, V., AND TRUSZCZYŃSKI, M. 2004. Ultimate approximation and its application in nonmonotonic knowledge representation systems. *Information and Computation* 192, 1, 84–121.
- EITER, T., IANNI, G., LUKASIEWICZ, T., SCHINDLAUER, R., AND TOMPITS, H. 2008. Combining answer set programming with description logics for the semantic web. *Artificial Intelligence* 172, 12–13, 1495–1539.
- EITER, T., LUKASIEWICZ, T., IANNI, G., AND SCHINDLAUER, R. 2011. Well-founded semantics for description logic programs in the semantic web. *ACM Transactions on Computational Logic* 12, 2, Article 3.
- LUKASIEWICZ, T. 2010. A novel combination of answer set programming with description logics for the semantic web. *IEEE TKDE* 22, 11, 1577–1592.
- MCCARTHY, J. 1980. Circumscription - a form of non-monotonic reasoning. *Artificial Intelligence* 13, 27–39, 171–172.
- MOTIK, B. AND ROSATI, R. 2010. Reconciling description logics and rules. *Journal of the ACM* 57, 5, 1–62.
- ROSATI, R. 2005. On the decidability and complexity of integrating ontologies and rules. *Journal of Web Semantics* 3, 1, 61–73.
- ROSATI, R. 2006. DL+log: Tight integration of description logics and disjunctive datalog. In *Proc. KR'06*. 68–78.
- SHEN, Y.-D. 2011. Well-supported semantics for description logic programs. In *Proc. IJCAI-11*. 1081–1086.
- SHEN, Y.-D. AND WANG, K. 2011. Extending logic programs with description logic expressions for the semantic web. In *Proc. International Semantic Web Conference*. 633–648.
- TARSKI, A. 1955. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics* 5:2, 285–309.
- YANG, Q., YOU, J.-H., AND FENG, Z. 2011. Integrating rules and description logics by circumscription. In *Proc. AAAI-11*.